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Lattice valued double fuzzy preproximity spaces

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ABSTRACT

The concept of lattice valued double fuzzy preproximity is introduced. The relationships among the double fuzzy preproximity, double fuzzy topology and double fuzzy interior (closure) operators are studied.

1. Introduction

Kubiak [1] and Šostak [2] introduced the notion of $(L-)$ fuzzy topological space as a generalization of L-topological spaces (originally called $(L-)$ fuzzy topological spaces by Chang [3] and Goguen [4]). It is the grade of openness of an L-fuzzy set. A general approach to the study of topological-type structures on fuzzy powersets was developed in [5–7,1,8].

As a generalization of fuzzy sets, the notion of intuitionistic fuzzy sets was introduced by Atanassov [9,10]. Recently, Çoker and his colleagues [11,12] introduced the notion of intuitionistic fuzzy topological space using intuitionistic fuzzy sets. Samanta and Mondal [13,14], introduced the notion of intuitionistic gradation of openness as a generalization of intuitionistic fuzzy topological spaces [12] and L-fuzzy topological spaces.

Working under the name "intuitionistic" did not continue because doubts were thrown about the suitability of this term, especially when working in the case of complete lattice L. These doubts were quickly ended in 2005 by Gutierrez Garcia and Rodabaugh [15]. They proved that this term is unsuitable in mathematics and applications. They concluded that they work under the name "double".

Fuzzifying preproximity spaces was studied by Ramadan et al. [16]. Lattice valued fuzzy preproximity was studied by Kim and Min [17] in 2005 and then double fuzzy preproximity was introduced and studied by Zahran and his coworkers [18,19]. In this paper, we give the definition of lattice valued double fuzzy preproximity spaces as an extension of L-fuzzy preproximity spaces introduced by Kim et al. [17] to the double fuzzy structure based on Girard monoids. Also, we investigate the relations among the double fuzzy preproximity, double fuzzy topology and double fuzzy interior (closure) operators.

2. Preliminaries

In our work two lattices will play the fundamental role. The first one is a frame, that is a complete lattice $L = (L, \leq, \wedge, \vee)$, satisfying the infinite distributivity law

$$a \wedge \left(\bigvee_{i\in I} b_i\right) = \bigvee_{i\in I}(a \wedge b_i), \quad \forall a \in L,\ \{b_i\}_{i\in I} \subset L.$$

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The top and the bottom element of L are denoted by 1_L and 0_L, respectively. Sometimes we will assume that the lattice L is equipped with one of the following operations: a monotone mapping $\star : L \longrightarrow L$ or a binary operation $\odot : L \times L \longrightarrow L$.

A lattice $L = (L, \leq, \wedge, \vee, \star)$ will be called adjunctive if the pair $(\star, \star)$ is an adjunction

$$(\star, \star) : L \vdash L^{\mathrm{op}}, \quad \text{i.e.}$$

$$a \leq b^\star \Longleftrightarrow b \leq a^\star, \quad \forall a, b \in L.$$

A lattice $L = (L, \leq, \wedge, \vee, \star)$ will be called involutive if $\star : L \longrightarrow L$ is an involution, i.e.

$$(a^\star)^\star = a, \quad \forall a \in L.$$

One can easily see that in an adjunctive involutive lattice involution $\star : L \longrightarrow L$ is order reversing:

$$a \leq b \Longrightarrow a^\star \geq b^\star, \quad \forall a, b \in L,$$

and conversely, if $\star : L \longrightarrow L$ is order reversing involution, then

$$(\star, \star) : L \vdash L^{\mathrm{op}}$$

is an adjunction.

A complete lattice $L = (L, \leq, \wedge, \vee)$ is called completely distributive, if it satisfies the following conditions: $\forall \{\{a_{i,j} : j \in J_i\} : i \in I\} \subset \wp(L) - \emptyset, I \neq \emptyset$,

$$(\mathrm{CD1}) \bigwedge_{i\in I} \left(\bigvee_{j\in J_i} a_{i,j} \right) = \bigvee_{\varphi \in \prod_{i\in I} J_i} \left(\bigwedge_{i\in I} a_{i,\varphi(i)} \right)$$

$$(\mathrm{CD2}) \bigvee_{i\in I} \left(\bigwedge_{j\in J_i} a_{i,j} \right) = \bigwedge_{\varphi \in \prod_{i\in I} J_i} \left(\bigvee_{i\in I} a_{i,\varphi(i)} \right).$$

Remark. Note that, since adjunctive involution is order reversing, complete completely distributive adjunctive lattices made the context for the approach to fuzzy topology develop by Hutton; see [20–22]. Further, such lattices, called Hutton lattices or Hutton algebras where used by many researchers.

Concerning the second, binary operation $\odot : L \times L \longrightarrow L$ (conjunction) it will be assumed that $L = (L, \leq, \wedge, \vee, \odot)$ is a commutative cl-monoid (see e.g. [23]), that is

(L1) $a \odot b = b \odot a$ for all $a, b \in L$.

(L2) $(a \odot b) \odot c = a \odot (b \odot c)$ for all $a, b, c \in L$.

(L3) $\odot$ is distributive over arbitrary joins:

$$a \odot \left(\bigvee_{i\in I} b_i \right) = \bigvee_{i\in I} (a \odot b_i), \quad \forall a \in L, \ \forall \{b_i\}_{i\in I} \subseteq L.$$

(L4) $a \odot 1_L = a$ and $a \odot 0_L = 0_L$, for all $a \in L$.

It is well known (see e.g. [23]) that in a cl-monoid there is a further binary operation $\longmapsto : L \longrightarrow L$ (residuation) which is related to conjunction $\odot$ by Galois connection:

$$a \odot b \leq c \Longleftrightarrow a \leq b \longmapsto c, \quad \forall a, b, c \in L.$$

Explicitly residuation is given by

$$a \longmapsto b = \bigvee \{c \in L | a \odot c \leq b\}.$$

One can easily see that residuation is nonincreasing by the first argument and nondecreasing by the second argument, and that $b \odot (b \longmapsto a) \leq a, \forall a, b \in L$. In particular $b \odot (b \longmapsto 0_L) \leq 0_L$ and hence

$$b \leq (b \longmapsto 0_L) \longmapsto 0_L, \quad \forall b \in L.$$

This allows to conclude, that by setting $a^\star = a \longmapsto 0_L$ we obtain an adjunction $(\star, \star) : L \vdash L^{\mathrm{op}}$. Indeed, if $a \leq (b \longmapsto 0_L)$, then

$$b \leq (b \longmapsto 0_L) \longmapsto 0_L \leq a \longmapsto 0_L.$$

A cl-monoid is called a Girard monoid [24] if

$$(a \longmapsto 0_L) \longmapsto 0_L = a, \quad \forall a \in L.$$

Hence in case L is a Girard monoid, residuation $\longmapsto$ induces an order reversing involution $\star : L \longrightarrow L$. In this paper, we assume that $L = (L, \leq, \odot, \oplus, \star)$ is a Girard monoid which is defined by

$$a \oplus b = (a^\star \odot b^\star)^\star.$$

Remark. (1) [25] Every completely distributive lattice $(L, \leq, \wedge, \vee, ')$ coincided with $\odot = \wedge$ is a commutative cl-monoid. In particular, the unit interval $([0, 1], \leq, \wedge, \vee, ')$, $(a' = 1 - a)$ is a Girard monoid.

(2) [26] Every continuous t-norm $([0, 1], \leq, t)$ coincided with $\odot = t$ is a commutative cl-monoid.

Lemma 2.1 ([17]). *Let $L = (L, \leq, \odot, \oplus, \star)$ be a Girard monoid. Then for each $a, b, c \in L$, $\{b_i\}_{i\in I} \subseteq L$, we have the following properties:*

(1) *If $a \leq b$, then $a \odot c \leq b \odot c$ and $a \oplus c \leq b \oplus c$.*
(2) *$a \odot b \leq a \wedge b \leq a \vee b \leq a \oplus b$.*
(3) *$a \oplus \left(\bigwedge_{i\in I} b_i\right) = \bigwedge_{i\in I}(a \oplus b_i)$.*

An important situation in our research will be the following. Let $L = (L, \leq, \wedge, \vee)$ be a lattice and X be a set. Then the L-powerset L^X becomes a lattice $(L^X, \leq, \wedge, \vee)$ by pointwise extending the lattice structure from L to L^X. Besides L^X is infinitely distributive whenever L was infinitely distributive. Moreover, if $L = (L, \leq, \wedge, \vee, \star)$ is an adjunctive (resp. involutive) lattice then by pointwise extending operation $\star$ from L to L^X, an adjunctive (resp. involutive)lattice $L^X = (L^X, \leq, \wedge, \vee, \star)$ is obtained. In case $L = (L, \leq, \wedge, \vee, \odot)$ is a cl-monoid, by pointwise extension of $\odot : L \times L \longrightarrow L$ to $\odot : L^X \times L^X \longrightarrow L^X$ we obtain a cl-monoid $L^X = (L^X, \leq, \wedge, \vee, \odot)$. Besides, if L is a Girard monoid, then L^X is a Girard monoid as well.

The second lattice belonging to the context of our work is denoted by M.

If $a \leq b$ or $b \leq a$ for each $a, b \in M$, then M is called a chain. A lattice M is called order dense, if for each $a, b \in M$ such that $a < b$, there exists $c \in M$ such that $a < c < b$. For each $\alpha \in M$, let $\underline{\alpha}$ denote the constant fuzzy subset of X with value α and $M_0 = M - \{0_M\}$, $M_1 = M - \{1_M\}$.

Definition 2.2 ([27]). The pair (τ, τ^*) of maps $\tau, \tau^* : L^X \longrightarrow M$ is called double fuzzy topology on X if it satisfies the following conditions:

(O1) $\tau(\lambda) \leq (\tau^*(\lambda) \mapsto 0_M)$, $\forall \lambda \in L^X$,
(O2) $\tau(\underline{0}) = \tau(\underline{1}) = 1_M$, $\tau^*(\underline{0}) = \tau^*(\underline{1}) = 0_M$,
(O3) $\tau(\lambda_1 \odot \lambda_2) \geq \tau(\lambda_1) \odot \tau(\lambda_2)$ and $\tau^*(\lambda_1 \odot \lambda_2) \leq \tau^*(\lambda_1) \oplus \tau^*(\lambda_2)$, for each $\lambda_1, \lambda_2 \in L^X$,
(O4) $\tau\left(\bigvee_{i\in\Gamma} \lambda_i\right) \geq \bigwedge_{i\in\Gamma} \tau(\lambda_i)$ and $\tau^*\left(\bigvee_{i\in\Gamma} \lambda_i\right) \leq \bigvee_{i\in\Gamma} \tau^*(\lambda_i)$, for each $\lambda_i \in L^X$, $i \in \Gamma$.

The triplet (X, τ, τ^*) is called a double fuzzy topological space. τ and τ^* may be interpreted as gradation of openness and gradation of nonopenness, respectively.

Definition 2.3 ([27]). Let (X, τ_1, τ_1^*) and (Y, τ_2, τ_2^*) be two double fuzzy topological spaces. A map $\phi : X \longrightarrow Y$ is called LF-continuous iff

$$\tau_1(\phi^{\leftarrow}(\mu)) \geq \tau_2(\mu)$$

and

$$\tau_1^*(\phi^{\leftarrow}(\mu)) \leq \tau_2^*(\mu), \quad \forall \mu \in L^Y.$$

The category of double fuzzy topological spaces and LF-continuous maps is denoted by DFTop.

Definition 2.4 ([28]). A map $\mathcal{I} : L^X \times M_0 \times M_1 \longrightarrow L^X$ is called a double fuzzy interior operator if it satisfies the following conditions: $\forall r \in M_0$, $s \in M_1$ such that $r \leq (s \longmapsto 0_M)$,

(I1) $\mathcal{I}(\underline{1}, r, s) = \underline{1}$.
(I2) $\mathcal{I}(\lambda, r, s) \leq \lambda$.
(I3) If $\lambda \leq \mu$, then $\mathcal{I}(\lambda, r, s) \leq \mathcal{I}(\mu, r, s)$.
(I4) If $r \leq r'$ and $s \geq s'$, then $\mathcal{I}(\lambda, r', s') \leq \mathcal{I}(\lambda, r, s)$.
(I5) $\mathcal{I}(\lambda \odot \mu, r \odot r', s \oplus s') \geq \mathcal{I}(\lambda, r, s) \odot \mathcal{I}(\mu, r', s')$.

The pair $(X, \mathcal{I})$ is called a double fuzzy interior space.

A double fuzzy interior space $(X, \mathcal{I})$ is called topological if

$$\mathcal{I}(\mathcal{I}(\lambda, r, s), r, s) = \mathcal{I}(\lambda, r, s), \quad \forall \lambda \in L^X,\ r \in M_0,\ s \in M_1 \text{ with } r \leq (s \longmapsto 0_M).$$

Definition 2.5 ([28]). Let $(X, \mathcal{I}_1)$ and $(Y, \mathcal{I}_2)$ be two double fuzzy interior spaces. A map $\phi : X \longrightarrow Y$ is called I-map iff

$$\phi^{\leftarrow}(\mathcal{I}_2(\mu, r, s)) \leq \mathcal{I}_1(\phi^{\leftarrow}(\mu), r, s), \quad \forall \mu \in L^Y,\ r \in M_0 \text{ and } s \in M_1.$$

The category of double fuzzy interior spaces and I-maps are denoted by DFInt.

Theorem 2.6 ([28]). *Let (X, τ, τ^*) be a double fuzzy topological space. For each $\lambda \in L^X$, $r \in M_0$ and $s \in M_1$ with $r \leq (s \mapsto 0_M)$, we define an operator $\mathcal{I}_{\tau,\tau^*} : L^X \times M_0 \times M_1 \longrightarrow L^X$ as follows:*

$$\mathcal{I}_{\tau,\tau^*}(\lambda, r, s) = \bigvee\{\mu \in L^X \mid \mu \leq \lambda, \tau(\mu) \geq r \text{ and } \tau^*(\mu) \leq s\}.$$

Then $(X, \mathcal{I}_{\tau,\tau^*})$ is a topological double fuzzy interior space and if $r = \bigvee\{r' \in M_0 \mid \mathcal{I}(\lambda, r', s') = \lambda\}$ and $s = \bigwedge\{s' \in M_1 \mid \mathcal{I}(\lambda, r', s') = \lambda\}$, then $\mathcal{I}(\lambda, r, s) = \lambda$.

Theorem 2.7 ([28]). *Let $(X, \mathcal{I})$ be a double fuzzy interior space. Define the mappings $\tau_{\mathcal{I}}, \tau^*_{\mathcal{I}} : L^X \longrightarrow M$ by*

$$\tau_{\mathcal{I}}(\lambda) = \bigvee\{r \in M_0 \mid \mathcal{I}(\lambda, r, s) = \lambda\}$$
$$\tau^*_{\mathcal{I}}(\lambda) = \bigwedge\{s \in M_1 \mid \mathcal{I}(\lambda, r, s) = \lambda\}.$$

Then:

(1) *The pair $(\tau_{\mathcal{I}}, \tau^*_{\mathcal{I}})$ is a double fuzzy topology on X.*
(2) *We have $\mathcal{I} = \mathcal{I}_{\tau_{\mathcal{I}},\tau^*_{\mathcal{I}}}$ iff double fuzzy interior space $(X, \mathcal{I})$ satisfies the following conditions:*
  (i) *It is topological.*
  (ii) *If $r = \bigvee\{r' \in M_0 \mid \mathcal{I}(\lambda, r', s') = \lambda\}$ and $s = \bigwedge\{s' \in M_1 \mid \mathcal{I}(\lambda, r', s') = \lambda\}$, then $\mathcal{I}(\lambda, r, s) = \lambda$.*

Corollary 2.8 ([28]). DFTop *is isomorphic to* DFInt.

Definition 2.9 ([27]). A map $\mathcal{C} : L^X \times M_0 \times M_1 \to L^X$ is called a double fuzzy closure operator on X if it satisfies the following conditions: $\forall r \in M_0, s \in M_1$ such that $r \le (s \longmapsto 0_M)$,
(C1) $\mathcal{C}(\underline{0}, r, s) = \underline{0}$, for all $r \in M_0, s \in M_1$.
(C2) $\mathcal{C}(\lambda, r, s) \ge \lambda$.
(C3) $\mathcal{C}(\lambda, r, s) \le \mathcal{C}(\mu, r, s)$, if $\lambda \le \mu$.
(C4) $\mathcal{C}(\lambda \oplus \mu, r \odot r', s \oplus s') \le \mathcal{C}(\lambda, r, s) \oplus \mathcal{C}(\mu, r', s')$,
(C5) $\mathcal{C}(\lambda, r, s) \le \mathcal{C}(\lambda, r', s')$, if $r \le r', s \ge s'$, where $r, r' \in M_0, s, s' \in M_1$.
The pair $(X, \mathcal{C})$ is called a double fuzzy closure space.
A double fuzzy closure operator $\mathcal{C}$ is called topological if
(C6) $\mathcal{C}(\mathcal{C}(\lambda, r, s), r, s) \le \mathcal{C}(\lambda, r, s)$.

Let $(X, \mathcal{C}_1)$ and $(Y, \mathcal{C}_2)$ be two double fuzzy closure spaces. A map $\phi : X \longrightarrow Y$ is called C-map iff

$$\phi^{\to}(\mathcal{C}_1(\lambda, r, s)) \le \mathcal{C}_2(\phi^{\to}(\lambda), r, s), \quad \forall \lambda \in L^X,\ r \in M_0 \text{ and } s \in M_1.$$

The category of double fuzzy closure spaces and C-maps are denoted by DFCl.

Theorem 2.10 ([27]). *Let (X, τ, τ^*) be a double fuzzy topological space. For each $\lambda \in L^X, r \in M_0, s \in M_1$ with $r \le (s \mapsto 0_M)$, we define an operator $\mathcal{C}_{\tau,\tau^*} : L^X \times M_0 \times M_1 \longrightarrow L^X$ as follows:*

$$\mathcal{C}_{\tau,\tau^*}(\lambda, r, s) = \bigwedge\{\mu \in L^X \mid \lambda \le \mu, \tau(\mu \mapsto \underline{0}) \ge r \text{ and } \tau^*(\mu \mapsto \underline{0}) \le s\}.$$

Then $(X, \mathcal{C}_{\tau,\tau^*})$ is a topological double fuzzy closure operator on X and if $r = \bigvee\{r' \in M_0 \mid \mathcal{C}(\lambda, r', s') = \lambda\}$ and $s = \bigwedge\{s' \in M_1 \mid \mathcal{C}(\lambda, r', s') = \lambda\}$, then $\mathcal{C}(\lambda, r, s) = \lambda$.

Lemma 2.11 ([28]). *Let $\mathcal{C}_{\tau,\tau^*}$ be a double fuzzy closure operator on X which defined as in Theorem 2.10. Then,*

$$\mathcal{C}_{\tau,\tau^*}(\lambda \longmapsto \underline{0}, r, s) = \mathcal{I}_{\tau,\tau^*}(\lambda, r, s) \longmapsto \underline{0}.$$

Theorem 2.12 ([27]). *Let $(X, \mathcal{C})$ be a double fuzzy closure space. Define the mappings $\tau_{\mathcal{C}}, \tau^*_{\mathcal{C}} : L^X \longrightarrow M$ on X by*

$$\tau_{\mathcal{C}}(\lambda) = \bigvee\{r \in M_0 \mid \mathcal{C}(\lambda \longmapsto \underline{0}, r, s) = \lambda \longmapsto \underline{0}\}$$
$$\tau^*_{\mathcal{C}}(\lambda) = \bigwedge\{s \in M_1 \mid \mathcal{C}(\lambda \longmapsto \underline{0}, r, s) = \lambda \longmapsto \underline{0}\}.$$

Then:

(1) *The pair $(\tau_{\mathcal{C}}, \tau^*_{\mathcal{C}})$ is a double fuzzy topology on X.*
(2) *We have $\mathcal{C} = \mathcal{C}_{\tau_{\mathcal{C}},\tau^*_{\mathcal{C}}}$ iff double fuzzy closure space $(X, \mathcal{C})$ satisfies the following conditions:*
  (i) *It is topological.*
  (ii) *If $r = \bigvee\{r' \in M_0 \mid \mathcal{C}(\lambda, r', s') = \lambda\}$ and $s = \bigwedge\{s' \in M_1 \mid \mathcal{C}(\lambda, r', s') = \lambda\}$, then $\mathcal{C}(\lambda, r, s) = \lambda$.*

Corollary 2.13 ([27]). DFTop *is isomorphic to* DFCl.

3. Lattice valued double fuzzy preproximity

Definition 3.1. The pair (δ, δ^*) of maps $\delta, \delta^* : L^X \times L^X \longrightarrow M$ is called a double fuzzy preproximity on X if it satisfies the following conditions:

(P1) $\delta(\lambda, \mu) \ge \delta^*(\lambda, \mu) \longmapsto 0_M$
(P2) $\delta(\underline{1}, \underline{0}) = \delta(\underline{0}, \underline{1}) = 0_M$ and $\delta^*(\underline{0}, \underline{1}) = \delta^*(\underline{1}, \underline{0}) = 1_M$

(P3) If $\delta(\lambda, \mu) \neq 1_M$ and $\delta^*(\lambda, \mu) \neq 0_M$, then $\lambda \leq \mu \longmapsto \underline{0}$

(P4) If $\lambda_1 \leq \lambda_2$, then $\delta(\lambda_1, \mu) \leq \delta(\lambda_2, \mu)$ and $\delta^*(\lambda_1, \mu) \geq \delta^*(\lambda_2, \mu)$.

(P5) $\delta(\lambda_1 \odot \lambda_2, \rho_1 \oplus \rho_2) \leq \delta(\lambda_1, \rho_1) \oplus \delta(\lambda_2, \rho_2)$ and $\delta^*(\lambda_1 \odot \lambda_2, \rho_1 \oplus \rho_2) \geq \delta^*(\lambda_1, \rho_1) \odot \delta^*(\lambda_2, \rho_2)$.

The triplet (X, δ, δ^*) is called a double fuzzy preproximity space. Also, we call $\delta(\lambda, \mu)$ a gradation of nearness and $\delta^*(\lambda, \mu)$ a gradation of non-nearness between λ and μ. A double fuzzy preproximity (δ, δ^*) is called a double fuzzy quasi-proximity if

(P6) $\delta(\lambda, \mu) \geq \bigwedge_{\nu \in L^X} \{\delta(\lambda, \nu) \oplus \delta(\nu \mapsto \underline{0}, \mu)\}$ and $\delta^*(\lambda, \mu) \leq \bigvee_{\nu \in L^X} \{\delta^*(\lambda, \nu) \odot \delta^*(\nu \mapsto \underline{0}, \mu)\}$.

A double fuzzy preproximity space is called principal provided that

(P7) $\delta\left(\bigvee_{i \in \Gamma} \lambda_i, \mu\right) \leq \bigvee_{i \in \Gamma} \delta(\lambda_i, \mu)$ and $\delta^*\left(\bigvee_{i \in \Gamma} \lambda_i, \mu\right) \geq \bigwedge_{i \in \Gamma} \delta^*(\lambda_i, \mu)$

A double fuzzy quasi-proximity is called double fuzzy proximity if

(P8) $\delta(\lambda, \mu) = \delta(\mu, \lambda)$ and $\delta^*(\lambda, \mu) = \delta^*(\mu, \lambda)$.

Remark. If $L = M = [0, 1]$, $\odot = \wedge$ and $\oplus = \vee$, the above definition coincides with that in the sense of Zahran and his coworkers [18].

Example 3.2. (1) Let X be a set. Define the maps $\delta, \delta^* : L^X \times L^X \longrightarrow M$ as follows:

$$\delta(\lambda, \mu) = \begin{cases} 0_M, & \text{if } \lambda = \underline{0} \text{ or } \mu = \underline{0} \\ 1_M, & \text{otherwise} \end{cases}$$

$$\delta^*(\lambda, \mu) = \begin{cases} 1_M, & \text{if } \lambda = \underline{0} \text{ or } \mu = \underline{0} \\ 0_M, & \text{otherwise.} \end{cases}$$

Since the other cases are easy. We will prove the condition (P5). Since $\lambda_1 \odot \lambda_2 \neq \underline{0}$ and $\mu_1 \oplus \mu_2 \neq \underline{0}$ imply $\lambda_1 \neq \underline{0}$, $\lambda_2 \neq \underline{0}$ and $\mu_1 \neq \underline{0}$ or $\mu_2 \neq \underline{0}$, we have

$$\delta(\lambda_1 \odot \lambda_2, \mu_1 \oplus \mu_2) \leq \delta(\lambda_1, \mu_1) \oplus \delta(\lambda_2, \mu_2)$$

and

$$\delta^*(\lambda_1 \odot \lambda_2, \mu_1 \oplus \mu_2) \geq \delta^*(\lambda_1, \mu_1) \odot \delta^*(\lambda_2, \mu_2).$$

Hence the triplet (X, δ, δ^*) is a principal double fuzzy proximity space.

(2) Let X be a set. Define the maps $\delta, \delta^* : L^X \times L^X \longrightarrow M$ as follows:

$$\delta(\lambda, \mu) = \begin{cases} 0_M, & \text{if } \lambda \leq \mu \longmapsto \underline{0} \\ 1_M, & \text{otherwise} \end{cases}$$

$$\delta^*(\lambda, \mu) = \begin{cases} 1_M, & \text{if } \lambda \leq \mu \longmapsto \underline{0} \\ 0_M, & \text{otherwise.} \end{cases}$$

Since the other cases are easy to see. Let us prove the conditions (P5) and (P6). Since $\lambda_1 \odot \lambda_2 \not\leq (\mu_1 \oplus \mu_2) \longmapsto \underline{0}$ implies $\lambda_1 \not\leq \mu_1 \longmapsto \underline{0}$ or $\lambda_2 \not\leq \mu_2 \longmapsto \underline{0}$, then we have

$$1_M = \delta(\lambda_1 \odot \lambda_2, \mu_1 \oplus \mu_2) = \delta(\lambda_1, \mu_1) \oplus \delta(\lambda_2, \mu_2)$$

and

$$0_M = \delta^*(\lambda_1 \odot \lambda_2, \mu_1 \oplus \mu_2) = \delta^*(\lambda_1, \mu_1) \odot \delta^*(\lambda_2, \mu_2).$$

So, condition (P5) is satisfied.

(P6) Since $\delta(\lambda, \mu) = 0_M$ and $\delta^*(\lambda, \mu) = 1_M$ imply $\lambda \leq \mu \longmapsto \underline{0}$, $\delta(\lambda, \lambda \longmapsto \underline{0}) = 0_M$, $\delta^*(\lambda, \lambda \longmapsto \underline{0}) = 1_M$ and $\delta(\lambda, \mu) = 0_M$, $\delta^*(\lambda, \mu) = 1_M$.

Therefore the triplet (X, δ, δ^*) is a double fuzzy quasi-proximity space.

Example 3.3. Let $X = \{x, y\}$ be a two point set and $L = M = [0, 1]$. Define binary operations $\odot, \oplus, \mapsto$ on $[0, 1]$ (called Lukasiewicz t-norm) by

$$x \odot y = \max\{x + y - 1, 0\}, \qquad x \oplus y = \min\{x + y, 1\}, \qquad x \mapsto y = \min\{1 - x + y, 1\}.$$

Define $\nu \in [0,1]^X$ as follows: $\nu(x) = 0.7$ and $\nu(y) = 0.6$. So, $(\nu \odot \nu)(x) = 0.4$ and $(\nu \odot \nu)(y) = 0.2$. Define maps $\delta, \delta^* : [0,1]^X \times [0,1]^X \longrightarrow [0,1]$ as follows:

$$\delta(\lambda,\mu) = \begin{cases} 0_M, & \text{if} \lambda = \underline{0} \text{ or } \mu = \underline{0} \\ \dfrac{1}{3}, & \text{if } \lambda \le \nu,\ \lambda \not\le (\nu \odot \nu),\ \underline{0} \neq \mu \le \nu \mapsto \underline{0} \\ \dfrac{1}{2}, & \text{if } \underline{0} \neq \lambda \le (\nu \odot \nu),\ \mu \le (\nu \odot \nu) \mapsto \underline{0},\ \mu \not\le \nu \mapsto \underline{0} \\ 1_M, & \text{otherwise} \end{cases}$$

$$\delta^*(\lambda,\mu) = \begin{cases} 1_M, & \text{if } \lambda = \underline{0} \text{ or } \mu = \underline{0} \\ \dfrac{2}{3}, & \text{if } \lambda \le \nu,\ \lambda \not\le (\nu \odot \nu),\ \underline{0} \neq \mu \le \nu \mapsto \underline{0} \\ \dfrac{1}{2}, & \text{if } \underline{0} \neq \lambda \le (\nu \odot \nu),\ \mu \le (\nu \odot \nu) \mapsto \underline{0},\ \mu \not\le \nu \mapsto \underline{0} \\ 0_M, & \text{otherwise.} \end{cases}$$

The triplet (X, δ, δ^*) is a lattice valued double fuzzy preproximity space because

$$\frac{1}{2} = \delta(\nu \odot \nu, (\nu \odot \nu) \mapsto \underline{0}) \le \delta(\nu, \nu \mapsto \underline{0}) \oplus \delta(\nu, \nu \mapsto \underline{0}) = \frac{2}{3}$$

and

$$\frac{1}{2} = \delta^*(\nu \odot \nu, (\nu \odot \nu) \mapsto \underline{0}) \ge \delta^*(\nu, \nu \mapsto \underline{0}) \odot \delta^*(\nu, \nu \mapsto \underline{0}) = \frac{1}{3}.$$

Definition 3.4. Let $(X, \delta_1, \delta_1^*)$ and $(Y, \delta_2, \delta_2^*)$ be two double fuzzy preproximity spaces. A map $\phi : (X, \delta_1, \delta_1^*) \longrightarrow (Y, \delta_2, \delta_2^*)$ is called double fuzzy preproximally continuous if

$$\delta_1(\lambda,\mu) \le \delta_2(\phi^{\rightarrow}(\lambda), \phi^{\rightarrow}(\mu))$$

and

$$\delta_1^*(\lambda,\mu) \ge \delta_2^*(\phi^{\rightarrow}(\lambda), \phi^{\rightarrow}(\mu)), \quad \forall \lambda, \mu \in L^X.$$

In this way, we obtain the category DFpp (resp. PDFpp) where the objects are double fuzzy preproximity (resp. principal double fuzzy preproximity) spaces and the morphisms are double fuzzy preproximally continuous maps between them.

Lemma 3.5. *Let $(X, \delta_1, \delta_1^*)$ and $(Y, \delta_2, \delta_2^*)$ be two double fuzzy preproximity spaces. Then the map $\phi : (X, \delta_1, \delta_1^*) \longrightarrow (Y, \delta_2, \delta_2^*)$ is double fuzzy preproximally continuous iff*

$$\delta_2(\nu,\rho) \ge \delta_1(\phi^{\leftarrow}(\nu), \phi^{\leftarrow}(\rho))$$

and

$$\delta_2^*(\nu,\rho) \le \delta_1^*(\phi^{\leftarrow}(\nu), \phi^{\leftarrow}(\rho)), \quad \forall \nu, \rho \in L^Y.$$

Clearly one equips all double fuzzy preproximities on a set X with partial ordering by (δ_1, δ_1^*) which is finer than (δ_2, δ_2^*) if the identity map of X is double fuzzy preproximally continuous from $(X, \delta_1, \delta_1^*)$ to $(Y, \delta_2, \delta_2^*)$, that is

$$\delta_1(\lambda,\mu) \le \delta_2(\lambda,\mu)$$

and

$$\delta_1^*(\lambda,\mu) \ge \delta_2^*(\lambda,\mu), \quad \forall \lambda, \mu \in L^X.$$

Theorem 3.6. *Let (X, δ, δ^*) be a principal double fuzzy preproximity space. Define the maps $\tau_\delta, \tau^*_{\delta^*} : L^X \longrightarrow M$ as follows:*

$$\tau_\delta(\lambda) = \delta(\lambda, \lambda \longmapsto \underline{0}) \longmapsto 0_M$$

and

$$\tau^*_{\delta^*}(\lambda) = \delta^*(\lambda, \lambda \longmapsto \underline{0}) \longmapsto 0_M$$

for each $\lambda \in L^X$. Then the pair $(\tau_\delta, \tau^*_{\delta^*})$ is a double fuzzy topology on X.

Proof. (O1) Since (P1), $\tau_\delta(\lambda) \le \tau^*_{\delta^*}(\lambda) \longmapsto 0_M$.

(O2) It is clear that $\tau_\delta(\underline{0}) = \tau_\delta(\underline{1}) = 1_M$ and $\tau^*_{\delta^*}(\underline{0}) = \tau^*_{\delta^*}(\underline{1}) = 0_M$.

(O3) It is easily proved from the following:

$$\begin{aligned}
\tau_\delta(\lambda_1 \odot \lambda_2) &= \delta(\lambda_1 \odot \lambda_2, (\lambda_1 \odot \lambda_2) \longmapsto \underline{0}) \longmapsto 0_M \\
&= \delta(\lambda_1 \odot \lambda_2, (\lambda_1 \longmapsto \underline{0}) \oplus (\lambda_2 \longmapsto \underline{0})) \longmapsto 0_M \\
&\ge (\delta(\lambda_1, \lambda_1 \longmapsto \underline{0}) \longmapsto 0_M) \odot (\delta(\lambda_2, \lambda_2 \longmapsto \underline{0}) \longmapsto 0_M) \\
&= \tau_\delta(\lambda_1) \odot \tau_\delta(\lambda_2).
\end{aligned}$$

$$\begin{aligned}
\tau^*_{\delta^*}(\lambda_1 \odot \lambda_2) &= \delta^*(\lambda_1 \odot \lambda_2, (\lambda_1 \odot \lambda_2) \longmapsto \underline{0}) \longmapsto 0_M \\
&= \delta^*(\lambda_1 \odot \lambda_2, (\lambda_1 \longmapsto \underline{0}) \oplus (\lambda_2 \longmapsto \underline{0})) \longmapsto 0_M \\
&\le (\delta^*(\lambda_1, \lambda_1 \longmapsto \underline{0}) \longmapsto 0_M) \oplus (\delta^*(\lambda_2, \lambda_2 \longmapsto \underline{0}) \longmapsto 0_M) \\
&= \tau^*_{\delta^*}(\lambda_1) \oplus \tau^*_{\delta^*}(\lambda_2).
\end{aligned}$$

(O4) For each family $\{\lambda_i\}_{i\in\Gamma} \subseteq L^X$, we have

$$\begin{aligned}
\tau_\delta\left(\bigvee_{i\in\Gamma} \lambda_i\right) &= \delta\left(\bigvee_{i\in\Gamma} \lambda_i, \left(\bigvee_{i\in\Gamma} \lambda_i\right) \longmapsto \underline{0}\right) \longmapsto 0_M \\
&= \delta\left(\bigvee_{i\in\Gamma} \lambda_i, \bigwedge_{i\in\Gamma} (\lambda_i \longmapsto \underline{0})\right) \longmapsto 0_M \\
&\ge \left(\bigvee_{i\in\Gamma} \delta\left(\lambda_i, \bigwedge_{i\in\Gamma} (\lambda_i \longmapsto \underline{0})\right)\right) \longmapsto 0_M \\
&\ge \left(\bigvee_{i\in\Gamma} \delta(\lambda_i, \lambda_i \longmapsto \underline{0})\right) \longmapsto 0_M \\
&= \bigwedge_{i\in\Gamma} (\delta(\lambda_i, \lambda_i \longmapsto \underline{0}) \longmapsto 0_M) = \bigwedge_{i\in\Gamma} \tau_\delta(\lambda_i).
\end{aligned}$$

$$\begin{aligned}
\tau^*_{\delta^*}\left(\bigvee_{i\in\Gamma} \lambda_i\right) &= \delta^*\left(\bigvee_{i\in\Gamma} \lambda_i, \left(\bigvee_{i\in\Gamma} \lambda_i\right) \longmapsto \underline{0}\right) \longmapsto 0_M \\
&= \delta^*\left(\bigvee_{i\in\Gamma} \lambda_i, \bigwedge_{i\in\Gamma} (\lambda_i \longmapsto \underline{0})\right) \longmapsto 0_M \\
&\le \left(\bigwedge_{i\in\Gamma} \delta^*\left(\lambda_i, \bigwedge_{i\in\Gamma} (\lambda_i \longmapsto \underline{0})\right)\right) \longmapsto 0_M \\
&= \bigvee_{i\in\Gamma} \delta^*\left(\lambda_i, \bigwedge_{i\in\Gamma} (\lambda_i \longmapsto \underline{0})\right) \longmapsto 0_M \\
&\le \bigvee_{i\in\Gamma} \delta^*(\lambda_i, \lambda_i \longmapsto \underline{0}) \longmapsto 0_M = \bigvee_{i\in\Gamma} \tau^*_{\delta^*}(\lambda_i). \quad \square
\end{aligned}$$

Theorem 3.7. *Let $\phi : (X, \delta_1, \delta_1^*) \longrightarrow (Y, \delta_2, \delta_2^*)$ be a double fuzzy preproximally continuous map. Then $\phi : (X, \tau_{\delta_1}, \tau^*_{\delta_1^*}) \longrightarrow (Y, \tau_{\delta_2}, \tau^*_{\delta_2^*})$ is LF-continuous with respect to the induced double fuzzy topological spaces.*

Proof. For each $\nu \in L^Y$,

$$\begin{aligned}
\tau_{\delta_1}(\phi^{\leftarrow}(\nu)) &= \delta_1(\phi^{\leftarrow}(\nu), \phi^{\leftarrow}(\nu) \longmapsto \underline{0}) \longmapsto 0_M \\
&\ge \delta_2(\nu, \nu \longmapsto \underline{0}) \longmapsto 0_M \\
&= \tau_{\delta_2}(\nu)
\end{aligned}$$

and

$$\begin{aligned}\tau^*_{\delta^*_1}(\phi^{\leftarrow}(\nu)) &= \delta^*_1(\phi^{\leftarrow}(\nu), \phi^{\leftarrow}(\nu) \longmapsto \underline{0}) \longmapsto 0_M \\ &\leq \delta^*_2(\nu, \nu \longmapsto \underline{0}) \longmapsto 0_M \\ &= \tau^*_{\delta^*_2}(\nu). \quad \square\end{aligned}$$

Theorem 3.8. *Let (X, δ, δ^*) be a double fuzzy preproximity space. Define a map $\mathcal{I}_{\delta,\delta^*} : L^X \times M_0 \times M_1 \longrightarrow L^X$ by*

$$\mathcal{I}_{\delta,\delta^*}(\lambda, r, s) = \bigvee\{\rho \in L^X \mid \delta(\rho, \lambda \mapsto \underline{0}) < r \mapsto 0_M \text{ and } \delta^*(\rho, \lambda \mapsto \underline{0}) > s \mapsto 0_M\}.$$

Then it satisfies the following properties:

(1) *The pair $(X, \mathcal{I}_{\delta,\delta^*})$ is a double fuzzy interior space.*
(2) *If (X, δ, δ^*) is a double fuzzy quasi-proximity space and L is a chain, then $(X, \mathcal{I}_{\delta,\delta^*})$ is topological.*

Proof. (1) (I1) Since $\delta(\underline{1}, \underline{0}) = 0_M$ and $\delta^*(\underline{1}, \underline{0}) = 1_M$, $\mathcal{I}_{\delta,\delta^*}(\underline{1}, r, s) = \underline{1}$.

(I2) Since $\delta(\rho, \lambda \mapsto \underline{0}) \neq 1_M$ and $\delta^*(\rho, \lambda \mapsto \underline{0}) \neq 0_M$, then by (P3), $\mathcal{I}_{\delta,\delta^*}(\lambda, r, s) \leq \lambda$.

(I3) For $\lambda \leq \mu$, since $\delta(\rho, \mu \mapsto \underline{0}) \leq \delta(\rho, \lambda \mapsto \underline{0}) < r \mapsto 0_M$ and $\delta^*(\rho, \mu \mapsto \underline{0}) \geq \delta^*(\rho, \lambda \mapsto \underline{0}) > s \mapsto 0_M$, we have

$$\mathcal{I}_{\delta,\delta^*}(\lambda, r, s) \leq \mathcal{I}_{\delta,\delta^*}(\mu, r, s).$$

(I4) For $r \leq r'$ and $s \geq s'$, since $\delta(\rho, \lambda \mapsto \underline{0}) < r' \mapsto 0_M \leq r \mapsto 0_M$ and $\delta^*(\rho, \lambda \mapsto \underline{0}) > s' \mapsto 0_M \geq s \mapsto 0_M$, we have

$$\mathcal{I}_{\delta,\delta^*}(\lambda, r', s') \leq \mathcal{I}_{\delta,\delta^*}(\lambda, r, s).$$

(I5) From (L3) and (P5), we have:

$$\begin{aligned}\mathcal{I}_{\delta,\delta^*}(\lambda, r, s) \odot \mathcal{I}_{\delta,\delta^*}(\mu, r', s') &= \left(\bigvee\{\nu \mid \delta(\nu, \lambda \mapsto \underline{0}) < r \mapsto 0_M \text{ and } \delta^*(\nu, \lambda \mapsto \underline{0}) > s \mapsto 0_M\}\right) \\ &\quad \odot \left(\bigvee\{\rho \mid \delta(\rho, \mu \mapsto \underline{0}) < r' \mapsto 0_M \text{ and } \delta^*(\rho, \mu \mapsto \underline{0}) > s' \mapsto 0_M\}\right) \\ &\leq \bigvee\{\nu \odot \rho \mid \delta(\nu \odot \rho, \lambda \mapsto \underline{0} \oplus \mu \mapsto \underline{0}) < (r \odot r') \mapsto 0_M \text{ and } \delta^*(\nu \odot \rho, \lambda \mapsto \underline{0} \oplus \mu \mapsto \underline{0}) > (s \oplus s') \mapsto 0_M\} \\ &\leq \bigvee\{\gamma \in L^X \mid \delta(\gamma, (\lambda \odot \mu) \mapsto \underline{0}) < (r \odot r') \mapsto 0_M \text{ and } \delta^*(\gamma, (\lambda \odot \mu) \mapsto \underline{0}) > (s \oplus s') \mapsto 0_M\} \\ &= \mathcal{I}_{\delta,\delta^*}(\lambda \odot \mu, r \odot r', s \oplus s').\end{aligned}$$

Hence $\mathcal{I}_{\delta,\delta^*}(\lambda \odot \mu, r \odot r', s \oplus s') \geq \mathcal{I}_{\delta,\delta^*}(\lambda, r, s) \odot \mathcal{I}_{\delta,\delta^*}(\mu, r', s')$.

(2) Let $\delta(\rho, \lambda \mapsto \underline{0}) < r \mapsto 0_M$ and $\delta^*(\rho, \lambda \mapsto \underline{0}) > s \mapsto 0_M$ be given. Since L is a chain and by (P6),

$$\bigwedge_{\nu \in L^X}\{\delta(\rho, \nu) \oplus \delta(\nu \mapsto \underline{0}, \lambda \mapsto \underline{0})\} \leq \delta(\rho, \lambda \mapsto \underline{0}) < r \mapsto 0_M$$
$$\bigvee_{\nu \in L^X}\{\delta^*(\rho, \nu) \odot \delta^*(\nu \mapsto \underline{0}, \lambda \mapsto \underline{0})\} \geq \delta^*(\rho, \lambda \mapsto \underline{0}) > s \mapsto 0_M,$$

there exists $\nu \in L^X$ such that

$$\delta(\rho, \nu) \oplus \delta(\nu \mapsto \underline{0}, \lambda \mapsto \underline{0}) < r \mapsto 0_M$$
$$\delta^*(\rho, \nu) \odot \delta^*(\nu \mapsto \underline{0}, \lambda \mapsto \underline{0}) > s \mapsto 0_M.$$

Hence,

$$\delta(\rho, \nu) < r \mapsto 0_M, \qquad \delta(\nu \mapsto \underline{0}, \lambda \mapsto \underline{0}) < r \mapsto 0_M$$

and

$$\delta^*(\rho, \nu) > s \mapsto 0_M, \qquad \delta^*(\nu \mapsto \underline{0}, \lambda \mapsto \underline{0}) > s \mapsto 0_M.$$

These imply $\mathcal{I}_{\delta,\delta^*}(\lambda, r, s) \geq \nu \mapsto \underline{0}$. Thus,

$$\delta(\rho, \mathcal{I}_{\delta,\delta^*}(\lambda, r, s) \mapsto \underline{0}) \leq \delta(\rho, \nu) < r \mapsto 0_M$$
$$\delta^*(\rho, \mathcal{I}_{\delta,\delta^*}(\lambda, r, s) \mapsto \underline{0}) \geq \delta^*(\rho, \nu) > s \mapsto 0_M.$$

Here, $\delta(\rho, \lambda \mapsto \underline{0}) < r \mapsto 0_M$, $\delta^*(\rho, \lambda \mapsto \underline{0}) > s \mapsto 0_M$ implies

$$\delta(\rho, \mathcal{I}_{\delta,\delta^*}(\lambda, r, s) \mapsto \underline{0}) < r \mapsto 0_M \quad \text{and} \quad \delta^*(\rho, \mathcal{I}_{\delta,\delta^*}(\lambda, r, s) \mapsto \underline{0}) > s \mapsto 0_M.$$

Therefore,

$$\begin{aligned}\mathcal{I}_{\delta,\delta^*}(\lambda, r, s) &= \bigvee\{\rho \mid \delta(\rho, \lambda \mapsto \underline{0}) < r \mapsto 0_M \text{ and } \delta^*(\rho, \lambda \mapsto \underline{0}) > s \mapsto 0_M\}\\ &\le \bigvee\{\rho \mid \delta(\rho, \mathcal{I}_{\delta,\delta^*}(\lambda, r, s) \mapsto \underline{0}) < r \mapsto 0_M, \delta^*(\rho, \mathcal{I}_{\delta,\delta^*}(\lambda, r, s) \mapsto \underline{0}) > s \mapsto 0_M\}\\ &= \mathcal{I}_{\delta,\delta^*}(\mathcal{I}_{\delta,\delta^*}(\lambda, r, s), r, s).\end{aligned}$$

Since $\mathcal{I}_{\delta,\delta^*}(\mathcal{I}_{\delta,\delta^*}(\lambda, r, s), r, s) \le \mathcal{I}_{\delta,\delta^*}(\lambda, r, s)$, $(X, \mathcal{I}_{\delta,\delta^*})$ is topological. □

Theorem 3.9. *Let (X, δ, δ^*) be a double fuzzy preproximity space. Define the mappings $\tau_{\mathcal{I}_{\delta,\delta^*}}, \tau^*_{\mathcal{I}_{\delta,\delta^*}} : L^X \longrightarrow M$ as follows:*

$$\tau_{\mathcal{I}_{\delta,\delta^*}}(\lambda) = \bigvee\{r \in M_0 \mid \mathcal{I}_{\delta,\delta^*}(\lambda, r, s) = \lambda\}$$
$$\tau^*_{\mathcal{I}_{\delta,\delta^*}}(\lambda) = \bigwedge\{s \in M_1 \mid \mathcal{I}_{\delta,\delta^*}(\lambda, r, s) = \lambda\}.$$

Then the pair $(\tau_{\mathcal{I}_{\delta,\delta^*}}, \tau^*_{\mathcal{I}_{\delta,\delta^*}})$ is a double fuzzy topology on X.

Proof. It is straightforward from Theorem 2.7. □

Theorem 3.10. *Let (X, δ, δ^*) be a double fuzzy preproximity space. Define a map $\mathcal{C}_{\delta,\delta^*} : L^X \times M_0 \times M_1 \longrightarrow L^X$ by*

$$\mathcal{C}_{\delta,\delta^*}(\lambda, r, s) = \bigwedge\{\mu \mapsto \underline{0} \mid \delta(\mu, \lambda) < r \mapsto 0_M \text{ and } \delta^*(\mu, \lambda) > s \mapsto 0_M\}.$$

Then $\mathcal{C}_{\delta,\delta^*}$ is a double fuzzy closure operator on X.

Theorem 3.11. *Let (X, δ, δ^*) be a double fuzzy preproximity space and let $\mathcal{C}_{\delta,\delta^*}$ be a double fuzzy closure operator induced by (δ, δ^*). Define the mappings $\tau_{\mathcal{C}_{\delta,\delta^*}}, \tau^*_{\mathcal{C}_{\delta,\delta^*}} : L^X \longrightarrow M$ as follows:*

$$\tau_{\mathcal{C}_{\delta,\delta^*}}(\lambda) = \bigvee\{r \in M_0 \mid \mathcal{C}_{\delta,\delta^*}(\lambda \mapsto \underline{0}, r, s) = \lambda \mapsto \underline{0}\}$$
$$\tau^*_{\mathcal{C}_{\delta,\delta^*}}(\lambda) = \bigwedge\{s \in M_1 \mid \mathcal{C}_{\delta,\delta^*}(\lambda \mapsto \underline{0}, r, s) = \lambda \mapsto \underline{0}\}.$$

Then the pair $(\tau_{\mathcal{C}_{\delta,\delta^*}}, \tau^*_{\mathcal{C}_{\delta,\delta^*}})$ is a double fuzzy topology on X.

Proof. It is straightforward from Theorems 2.12 and 3.10. □

Theorem 3.12. *Let $(X, \delta_1, \delta_1^*)$ and $(Y, \delta_2, \delta_2^*)$ be two double fuzzy preproximity spaces. If $\phi : (X, \delta_1, \delta_1^*) \longrightarrow (Y, \delta_2, \delta_2^*)$ is double fuzzy preproximally continuous, then:*

(1) *$\phi : (X, \mathcal{C}_{\delta_1,\delta_1^*}) \longrightarrow (Y, \mathcal{C}_{\delta_2,\delta_2^*})$ is C-map.*
(2) *$\mathcal{C}_{\delta_1,\delta_1^*}(\phi^{\leftarrow}(\nu), r, s) \le \phi^{\leftarrow}(\mathcal{C}_{\delta_2,\delta_2^*}(\nu, r, s))$ for each $\nu \in L^Y$.*
(3) *$\phi : (X, \tau_{\mathcal{C}_{\delta_1,\delta_1^*}}, \tau^*_{\mathcal{C}_{\delta_1,\delta_1^*}}) \longrightarrow (Y, \tau_{\mathcal{C}_{\delta_2,\delta_2^*}}, \tau^*_{\mathcal{C}_{\delta_2,\delta_2^*}})$ is LF-continuous.*
(4) *$\phi : (X, \mathcal{I}_{\delta_1,\delta_1^*}) \longrightarrow (Y, \mathcal{I}_{\delta_2,\delta_2^*})$ is I-map.*

Proof. (1) Let $\lambda \in L^X$, $r \in M_0$ and $s \in M_1$ with $r \le s \mapsto 0_M$, then

$$\begin{aligned}\phi^{\leftarrow}(\mathcal{C}_{\delta_2,\delta_2^*}(\phi^{\rightarrow}(\lambda), r, s)) &= \phi^{\leftarrow}\left(\bigwedge\{\mu \mapsto \underline{0} \mid \delta_2(\mu, \phi^{\rightarrow}(\lambda)) < r \mapsto 0_M, \delta_2^*(\mu, \phi^{\rightarrow}(\lambda)) > s \mapsto 0_M\}\right)\\ &= \bigwedge\{\phi^{\leftarrow}(\mu \mapsto \underline{0}) \mid \delta_2(\mu, \phi^{\rightarrow}(\lambda)) < r \mapsto 0_M, \delta_2^*(\mu, \phi^{\rightarrow}(\lambda)) > s \mapsto 0_M\}\\ &\ge \bigwedge\{\phi^{\leftarrow}(\mu) \mapsto \underline{0} \mid \delta_1(\phi^{\leftarrow}(\mu), \phi^{\leftarrow}(\phi^{\rightarrow}(\lambda))) < r \mapsto 0_M \text{ and } \delta_1^*(\phi^{\leftarrow}(\mu), \phi^{\leftarrow}(\phi^{\rightarrow}(\lambda))) > s \mapsto 0_M\}\\ &\ge \bigwedge\{\phi^{\leftarrow}(\mu) \mapsto \underline{0} \mid \delta_1(\phi^{\leftarrow}(\mu), \lambda) < r \mapsto 0_M, \delta_1^*(\phi^{\leftarrow}(\mu), \lambda) > s \mapsto 0_M\}\\ &= \mathcal{C}_{\delta_1,\delta_1^*}(\lambda, r, s).\end{aligned}$$

Therefore,

$$\phi^{\rightarrow}(\mathcal{C}_{\delta_1,\delta_1^*}(\lambda, r, s)) \le \phi^{\rightarrow}\phi^{\leftarrow}(\mathcal{C}_{\delta_2,\delta_2^*}(\phi^{\rightarrow}(\lambda), r, s)) \le \mathcal{C}_{\delta_2,\delta_2^*}(\phi^{\rightarrow}(\lambda), r, s).$$

(2) For each $\nu \in L^Y$, put $\lambda = \phi^{\leftarrow}(\nu)$, from (1)

$$\phi^{\rightarrow}(\mathcal{C}_{\delta_1,\delta_1^*}(\phi^{\leftarrow}(\nu), r, s)) \le \mathcal{C}_{\delta_2,\delta_2^*}(\phi^{\rightarrow}\phi^{\leftarrow}(\nu), r, s) \le \mathcal{C}_{\delta_2,\delta_2^*}(\nu, r, s).$$

Since $\phi^{\rightarrow}\phi^{\leftarrow}(\nu) \le \nu$. Hence

$$\mathcal{C}_{\delta_1,\delta_1^*}(\phi^{\leftarrow}(\nu), r, s) \le \phi^{\leftarrow}(\mathcal{C}_{\delta_2,\delta_2^*}(\nu, r, s)), \quad \forall \nu \in L^Y.$$

(3) Let $\mu \in L^Y$,

$$\begin{aligned}\tau_{\mathcal{C}_{\delta_2,\delta_2^*}}(\mu) &= \bigvee\{r \in M_0 \mid \mathcal{C}_{\delta_2,\delta_2^*}(\mu \mapsto \underline{0}, r, s) = \mu \mapsto \underline{0}\}\\ &\le \bigvee\{r \in M_0 \mid \phi^{\leftarrow}(\mathcal{C}_{\delta_2,\delta_2^*}(\mu \mapsto \underline{0}, r, s)) = \phi^{\leftarrow}(\mu \mapsto \underline{0})\}\\ &\le \bigvee\{r \in M_0 \mid \mathcal{C}_{\delta_1,\delta_1^*}(\phi^{\leftarrow}(\mu \mapsto \underline{0}), r, s) = \phi^{\leftarrow}(\mu \mapsto \underline{0})\}\\ &= \tau_{\mathcal{C}_{\delta_1,\delta_1^*}}(\phi^{\leftarrow}(\mu)).\end{aligned}$$

Similarly,

$$\tau^*_{\mathcal{C}_{\delta_2,\delta_2^*}}(\mu) \ge \tau^*_{\mathcal{C}_{\delta_1,\delta_1^*}}(\phi^{\leftarrow}(\mu)).$$

(4) Let $\mu \in L^Y$,

$$\begin{aligned}\mathcal{I}_{\delta_1,\delta_1^*}(\phi^{\leftarrow}(\mu), r, s) \mapsto \underline{0} &= \mathcal{C}_{\delta_1,\delta_1^*}(\phi^{\leftarrow}(\mu \mapsto \underline{0}), r, s)\\ &\le \phi^{\leftarrow}\phi^{\rightarrow}(\mathcal{C}_{\delta_1,\delta_1^*}(\phi^{\leftarrow}(\mu \mapsto \underline{0}), r, s))\\ &\le \phi^{\leftarrow}(\mathcal{C}_{\delta_2,\delta_2^*}(\phi^{\rightarrow}\phi^{\leftarrow}(\mu \mapsto \underline{0}), r, s))\\ &\le \phi^{\leftarrow}(\mathcal{C}_{\delta_2,\delta_2^*}(\mu \mapsto \underline{0}, r, s)).\end{aligned}$$

Hence

$$\mathcal{I}_{\delta_1,\delta_1^*}(\phi^{\leftarrow}(\mu), r, s) \ge \phi^{\leftarrow}(\mathcal{C}_{\delta_2,\delta_2^*}(\mu \mapsto \underline{0}, r, s) \mapsto \underline{0}) = \phi^{\leftarrow}(\mathcal{I}_{\delta_2,\delta_2^*}(\mu, r, s)). \quad \square$$

Theorem 3.13. *Let (X, τ, τ^*) be a double fuzzy topological space. Define the maps $\delta_\tau, \delta^*_{\tau^*} : L^X \times L^X \longrightarrow M$ as follows:*

$$\delta_\tau(\lambda, \mu) = \begin{cases} \bigwedge\{\tau(\nu) \mapsto 0_M \mid \nu \in \Theta_{\lambda,\mu}\}, & \text{if } \Theta_{\lambda,\mu} \neq \emptyset,\\ 1_M, & \text{if } \Theta_{\lambda,\mu} = \emptyset.\end{cases}$$

and

$$\delta^*_{\tau^*}(\lambda, \mu) = \begin{cases} \bigvee\{\tau^*(\nu) \mapsto 0_M \mid \nu \in \Theta_{\lambda,\mu}\}, & \text{if } \Theta_{\lambda,\mu} \neq \emptyset,\\ 0_M, & \text{if } \Theta_{\lambda,\mu} = \emptyset\end{cases}$$

where $\Theta_{\lambda,\mu} = \{\nu \mid \lambda \le \nu \le \mu \mapsto \underline{0}\}$. Then we have the following properties:

(1) The pair $(\delta_\tau, \delta^*_{\tau^*})$ is a double fuzzy quasi-proximity on X.
(2) If M is a completely distributive lattice, then $(\delta_\tau, \delta^*_{\tau^*})$ is principal.
(3) If (δ, δ^*) is a principal double fuzzy preproximity on X, then $\delta \le \delta_{\tau_\delta}$ and $\delta^* \ge \delta^*_{\tau^*_{\delta^*}}$.
(4) If M is an order dense chain, then $\tau_{\mathcal{I}_{\delta_\tau,\delta^*_{\tau^*}}} = \tau$, $\tau^*_{\mathcal{I}_{\delta_\tau,\delta^*_{\tau^*}}} = \tau^*$.
(5) If M is a completely distributive lattice, then $\tau_{\delta_\tau} = \tau$ and $\tau^*_{\delta^*_{\tau^*}} = \tau^*$.

Proof. (1) (P1) $\delta_\tau(\lambda, \mu) \ge \delta^*_{\tau^*}(\lambda, \mu) \mapsto 0_M$. Since

$$\begin{aligned}\bigwedge\{\tau(\nu) \mapsto 0_M \mid \nu \in \Theta_{\lambda,\mu}\} &\ge \bigwedge\{(\tau^*(\nu) \mapsto 0_M) \mapsto 0_M \mid \nu \in \Theta_{\lambda,\mu}\}\\ &= \left(\bigvee\{\tau^*(\nu) \mapsto 0_M \mid \nu \in \Theta_{\lambda,\mu}\}\right) \mapsto 0_M.\end{aligned}$$

(P2) It is trivial from definitions, $\delta_\tau(\underline{0}, \underline{1}) = \delta_\tau(\underline{1}, \underline{0}) = 0_M$ and $\delta^*_{\tau^*}(\underline{0}, \underline{1}) = \delta^*_{\tau^*}(\underline{1}, \underline{0}) = 1_M$.
(P3) $\delta_\tau(\lambda, \mu) \neq 1_M$ and $\delta^*_{\tau^*}(\lambda, \mu) \neq 0_M$, then by definition $\Theta_{\lambda,\mu} \neq \emptyset$. Hence $\lambda \le \mu \mapsto \underline{0}$.
(P4) Let $\lambda_1 \le \lambda_2$. Then we have $\Theta_{\lambda_2,\mu} \subseteq \Theta_{\lambda_1,\mu}$. In this case, $\delta_\tau(\lambda_1, \mu) \le \delta_\tau(\lambda_2, \mu)$ and $\delta^*_{\tau^*}(\lambda_1, \mu) \ge \delta^*_{\tau^*}(\lambda_2, \mu)$.
(P5) If $\Theta_{\lambda_1,\rho_1} = \emptyset$ or $\Theta_{\lambda_2,\rho_2} = \emptyset$, then

$$\delta_\tau(\lambda_1 \odot \lambda_2, \rho_1 \oplus \rho_2) \le \delta_\tau(\lambda_1, \rho_1) \oplus \delta_\tau(\lambda_2, \rho_2) = 1_M.$$
$$\delta^*_{\tau^*}(\lambda_1 \odot \lambda_2, \rho_1 \oplus \rho_2) \ge \delta^*_{\tau^*}(\lambda_1, \rho_1) \odot \delta^*_{\tau^*}(\lambda_2, \rho_2) = 0_M.$$

Let $\Theta_{\lambda_1,\rho_1} \neq \emptyset$ and $\Theta_{\lambda_2,\rho_2} \neq \emptyset$. For all $\nu_i \in L^X$ with $\lambda_i \leq \nu_i \leq \rho_i \mapsto \underline{0}$, $i = 1, 2$, we have $\lambda_1 \odot \lambda_2 \leq (\nu_1 \odot \nu_2) \leq (\rho_1 \oplus \rho_2) \mapsto \underline{0}$ such that

$$\begin{aligned}
\delta_\tau(\lambda_1,\rho_1) \oplus \delta_\tau(\lambda_2,\rho_2) &= \left(\bigwedge\{\tau(\nu_1) \mapsto 0_M \mid \nu_1 \in \Theta_{\lambda_1,\rho_1}\}\right) \oplus \left(\bigwedge\{\tau(\nu_2) \mapsto 0_M \mid \nu_2 \in \Theta_{\lambda_2,\rho_2}\}\right) \\
&= \bigwedge\{(\tau(\nu_1) \mapsto 0_M) \oplus (\tau(\nu_2) \mapsto 0_M) \mid \nu_1 \in \Theta_{\lambda_1,\rho_1}, \nu_2 \in \Theta_{\lambda_2,\rho_2}\} \\
&\geq \bigwedge\{(\tau(\nu_1) \odot \tau(\nu_2)) \mapsto 0_M \mid \nu_1 \odot \nu_2 \in \Theta_{\lambda_1\odot\lambda_2,\rho_1\oplus\rho_2}\} \\
&\geq \bigwedge\{\tau(\nu_1 \odot \nu_2) \mapsto 0_M \mid \nu_1 \odot \nu_2 \in \Theta_{\lambda_1\odot\lambda_2,\rho_1\oplus\rho_2}\} \\
&\geq \bigwedge\{\tau(\nu) \mapsto 0_M \mid \nu \in \Theta_{\lambda_1\odot\lambda_2,\rho_1\oplus\rho_2}\} \\
&= \delta_\tau(\lambda_1 \odot \lambda_2, \rho_1 \oplus \rho_2).
\end{aligned}$$

$$\begin{aligned}
\delta^*_{\tau^*}(\lambda_1,\rho_1) \odot \delta^*_{\tau^*}(\lambda_2,\rho_2) &= \left(\bigvee\{\tau^*(\nu_1) \mapsto 0_M \mid \nu_1 \in \Theta_{\lambda_1,\rho_1}\}\right) \odot \left(\bigvee\{\tau^*(\nu_2) \mapsto 0_M \mid \nu_2 \in \Theta_{\lambda_2,\rho_2}\}\right) \\
&= \bigvee\{(\tau^*(\nu_1) \mapsto 0_M) \odot (\tau^*(\nu_2) \mapsto 0_M) \mid \nu_1 \in \Theta_{\lambda_1,\rho_1}, \nu_2 \in \Theta_{\lambda_2,\rho_2}\} \\
&\leq \bigvee\{(\tau^*(\nu_1) \oplus \tau^*(\nu_2)) \mapsto 0_M \mid \nu_1 \odot \nu_2 \in \Theta_{\lambda_1\odot\lambda_2,\rho_1\oplus\rho_2}\} \\
&\leq \bigwedge\{\tau^*(\nu_1 \odot \nu_2) \mapsto 0_M \mid \nu_1 \odot \nu_2 \in \Theta_{\lambda_1\odot\lambda_2,\rho_1\oplus\rho_2}\} \\
&\leq \bigvee\{\tau^*(\nu) \mapsto 0_M \mid \nu \in \Theta_{\lambda_1\odot\lambda_2,\rho_1\oplus\rho_2}\} \\
&= \delta^*_{\tau^*}(\lambda_1 \odot \lambda_2, \rho_1 \oplus \rho_2).
\end{aligned}$$

Hence,

$$\delta_\tau(\lambda_1 \odot \lambda_2, \rho_1 \oplus \rho_2) \leq \delta_\tau(\lambda_1,\rho_1) \oplus \delta_\tau(\lambda_2,\rho_2)$$

and

$$\delta^*_{\tau^*}(\lambda_1 \odot \lambda_2, \rho_1 \oplus \rho_2) \geq \delta^*_{\tau^*}(\lambda_1,\rho_1) \odot \delta^*_{\tau^*}(\lambda_2,\rho_2).$$

(P6) If $\Theta_{\lambda,\mu} = \emptyset$, it is straightforward. Let $\Theta_{\lambda,\mu} \neq \emptyset$. Since

$$\begin{aligned}
&\tau(\nu) \mapsto 0_M \geq \bigwedge\{(\tau(\rho) \mapsto 0_M) \oplus (\tau(\gamma) \mapsto 0_M) \mid \rho \in \Theta_{\lambda,\nu\mapsto\underline{0}}, \gamma \in \Theta_{\nu,\mu}\}, \\
&\tau^*(\nu) \mapsto 0_M \leq \bigvee\{(\tau^*(\rho) \mapsto 0_M) \odot (\tau^*(\gamma) \mapsto 0_M) \mid \rho \in \Theta_{\lambda,\nu\mapsto\underline{0}}, \gamma \in \Theta_{\nu,\mu}\}.
\end{aligned}$$

We have:

$$\begin{aligned}
\delta_\tau(\lambda,\mu) &= \bigwedge\{\tau(\nu) \mapsto 0_M \mid \nu \in \Theta_{\lambda,\mu}\} \\
&= \bigwedge\{\tau(\nu) \mapsto 0_M \mid \nu \in \Theta_{\lambda,\nu\mapsto\underline{0}}, \nu \in \Theta_{\nu,\mu}\} \\
&\geq \bigwedge_\nu \left\{\bigwedge\{(\tau(\rho) \mapsto 0_M) \oplus (\tau(\gamma) \mapsto 0_M) \mid \rho \in \Theta_{\lambda,\nu\mapsto\underline{0}}, \gamma \in \Theta_{\nu,\mu}\}\right\} \\
&\geq \bigwedge_\nu \left\{\left(\bigwedge\{\tau(\rho) \mapsto 0_M \mid \rho \in \Theta_{\lambda,\nu\mapsto\underline{0}}\}\right) \oplus \left(\bigwedge\{\tau(\gamma) \mapsto 0_M \mid \gamma \in \Theta_{\nu,\mu}\}\right)\right\} \\
&= \bigwedge_{\nu\in L^X} (\delta_\tau(\lambda, \nu \mapsto \underline{0}) \oplus \delta_\tau(\nu,\mu))
\end{aligned}$$

$$\begin{aligned}
\delta^*_{\tau^*}(\lambda,\mu) &= \bigvee\{\tau^*(\nu)\mapsto 0_M \mid \nu\in\Theta_{\lambda,\mu}\}\\
&= \bigvee\{\tau^*(\nu)\mapsto 0_M \mid \nu\in\Theta_{\lambda,\nu\mapsto\underline{0}},\ \nu\in\Theta_{\nu,\mu}\}\\
&\le \bigvee_{\nu}\left\{\bigvee\{(\tau^*(\rho)\mapsto 0_M)\odot(\tau^*(\gamma)\mapsto 0_M)\mid \rho\in\Theta_{\lambda,\nu\mapsto\underline{0}},\ \gamma\in\Theta_{\nu,\mu}\}\right\}\\
&\le \bigvee_{\nu}\left\{\left(\bigvee\{\tau^*(\rho)\mapsto 0_M\mid \rho\in\Theta_{\lambda,\nu\mapsto\underline{0}}\}\right)\odot\left(\bigvee\{\tau^*(\gamma)\mapsto 0_M\mid \gamma\in\Theta_{\nu,\mu}\}\right)\right\}\\
&= \bigvee_{\nu\in L^X}(\delta^*_{\tau^*}(\lambda,\nu\mapsto\underline{0})\odot\delta^*_{\tau^*}(\nu,\mu)).
\end{aligned}$$

(2) (P7) For all $\nu_i\in L^X$ with $\lambda_i\le\nu_i\le\mu\mapsto\underline{0}$, we have $\bigvee_{i\in\Gamma}\lambda_i\le\bigvee_{i\in\Gamma}\nu_i\le\mu\mapsto\underline{0}$,

$$\delta_\tau\left(\bigvee_{i\in\Gamma}\lambda_i,\mu\right)\le\tau\left(\bigvee_{i\in\Gamma}\nu_i\right)\mapsto 0_M\le\left(\bigwedge_{i\in\Gamma}\tau(\nu_i)\right)\mapsto 0_M=\bigvee_{i\in\Gamma}(\tau(\nu_i)\mapsto 0_M)$$
$$\delta^*_{\tau^*}\left(\bigvee_{i\in\Gamma}\lambda_i,\mu\right)\ge\tau^*\left(\bigvee_{i\in\Gamma}\nu_i\right)\mapsto 0_M\ge\left(\bigvee_{i\in\Gamma}\tau^*(\nu_i)\right)\mapsto 0_M=\bigwedge_{i\in\Gamma}(\tau^*(\nu_i)\mapsto 0_M).$$

Hence

$$\begin{aligned}
\bigvee_{i\in\Gamma}\delta_\tau(\lambda_i,\mu) &= \bigvee_{i\in\Gamma}\left(\bigwedge\{\tau(\nu_i)\mapsto 0_M\mid \nu_i\in\Theta_{\lambda_i,\mu}\}\right)\\
&= \bigwedge\left(\bigvee_{i\in\Gamma}\{\tau(\nu_i)\mapsto 0_M\mid \nu_i\in\Theta_{\lambda_i,\mu}\}\right)\\
&\ge \bigwedge\left\{\tau\left(\bigvee_{i\in\Gamma}\nu_i\right)\mapsto 0_M\,\middle|\,\bigvee_{i\in\Gamma}\nu_i\in\Theta_{\vee_{i\in\Gamma}\lambda_i,\mu}\right\}\\
&\ge \delta_\tau\left(\bigvee_{i\in\Gamma}\lambda_i,\mu\right).\\
\bigwedge_{i\in\Gamma}\delta^*_{\tau^*}(\lambda_i,\mu) &= \bigwedge_{i\in\Gamma}\left(\bigvee\{\tau^*(\nu_i)\mapsto 0_M\mid \nu_i\in\Theta_{\lambda_i,\mu}\}\right)\\
&= \bigvee\left(\bigwedge_{i\in\Gamma}\{\tau^*(\nu_i)\mapsto 0_M\mid \nu_i\in\Theta_{\lambda_i,\mu}\}\right)\\
&\le \bigvee\left\{\tau^*\left(\bigvee_{i\in\Gamma}\nu_i\right)\mapsto 0_M\,\middle|\,\bigvee_{i\in\Gamma}\nu_i\in\Theta_{\vee_{i\in\Gamma}\lambda_i,\mu}\right\}\\
&\le \delta^*_{\tau^*}\left(\bigvee_{i\in\Gamma}\lambda_i,\mu\right).
\end{aligned}$$

(3) Since $\delta(\lambda,\mu)\le\delta(\nu,\nu\mapsto\underline{0})$, for $\lambda\le\nu\le\mu\mapsto\underline{0}$ we have:

$$\begin{aligned}
\delta_{\tau_\delta}(\lambda,\mu) &= \bigwedge\{\tau_\delta(\nu)\mapsto 0_M\mid \lambda\le\nu\le\mu\mapsto\underline{0}\}\\
&= \bigwedge\{\delta(\nu,\nu\mapsto\underline{0})\mid \lambda\le\nu\le\mu\mapsto\underline{0}\}\\
&\ge \delta(\lambda,\mu),
\end{aligned}$$

$$\begin{aligned}\delta^*_{\tau^*_{\delta^*}}(\lambda,\mu) &= \bigvee\{\tau^*_{\delta^*}(\nu)\mapsto 0_M \mid \lambda\le\nu\le\mu\mapsto \underline{0}\}\\ &= \bigvee\{\delta^*(\nu,\nu\mapsto\underline{0}) \mid \lambda\le\nu\le\mu\mapsto\underline{0}\}\\ &\le \delta^*(\lambda,\mu).\end{aligned}$$

(4) Suppose $\tau_{\mathcal{I}_{\delta_\tau,\delta^*_{\tau^*}}} \not\ge \tau$ and $\tau^*_{\mathcal{I}_{\delta_\tau,\delta^*_{\tau^*}}} \not\le \tau^*$. Since M is an order dense chain, there exists $\lambda\in L^X$, $r\in M_0$ and $s\in M_1$ such that

$$\tau_{\mathcal{I}_{\delta_\tau,\delta^*_{\tau^*}}}(\lambda) < r < \tau(\lambda) \quad\text{and}\quad \tau^*_{\mathcal{I}_{\delta_\tau,\delta^*_{\tau^*}}}(\lambda) > s > \tau^*(\lambda).$$

Since $\tau(\lambda) > r$, $\tau^*(\lambda) < s$, we have

$$\delta_\tau(\lambda,\lambda\mapsto\underline{0}) = \tau(\lambda)\mapsto 0_M < r\mapsto 0_M$$

and

$$\delta^*_{\tau^*}(\lambda,\lambda\mapsto\underline{0}) = \tau^*(\lambda)\mapsto 0_M > s\mapsto 0_M.$$

So, $\mathcal{I}_{\delta_\tau,\delta^*_{\tau^*}}(\lambda,r,s) \ge \lambda$. Therefore, $\tau_{\mathcal{I}_{\delta_\tau,\delta^*_{\tau^*}}}(\lambda) \ge r$ and $\tau^*_{\mathcal{I}_{\delta_\tau,\delta^*_{\tau^*}}}(\lambda) \le s$. This is a contradiction. Hence $\tau_{\mathcal{I}_{\delta_\tau,\delta^*_{\tau^*}}} \ge \tau$ and $\tau^*_{\mathcal{I}_{\delta_\tau,\delta^*_{\tau^*}}} \le \tau^*$.

Suppose $\tau_{\mathcal{I}_{\delta_\tau,\delta^*_{\tau^*}}} \not\le \tau$ and $\tau^*_{\mathcal{I}_{\delta_\tau,\delta^*_{\tau^*}}} \not\ge \tau^*$. Since M is an order dense chain, there exists $\lambda\in L^X$ with $\mathcal{I}_{\delta_\tau,\delta^*_{\tau^*}}(\lambda,r,s)=\lambda$ such that

$$\tau_{\mathcal{I}_{\delta_\tau,\delta^*_{\tau^*}}}(\lambda) \ge r > \tau(\lambda) \quad\text{and}\quad \tau^*_{\mathcal{I}_{\delta_\tau,\delta^*_{\tau^*}}}(\lambda) \le s < \tau^*(\lambda).$$

Since

$$\lambda = \mathcal{I}_{\delta_\tau,\delta^*_{\tau^*}}(\lambda,r,s) = \bigvee\{\rho_i \mid \delta_\tau(\rho_i,\lambda\mapsto\underline{0}) < r\mapsto 0_M,\ \delta^*_{\tau^*}(\rho_i,\lambda\mapsto\underline{0}) > s\mapsto 0_M\}$$

by the definition of $(\delta_\tau,\delta^*_{\tau^*})$, for all i, there exists ν_i with $\rho_i\le\nu_i\le\lambda$ such that $\tau(\nu_i) > r$ and $\tau^*(\nu_i) < s$. Thus $\lambda=\bigvee_i\rho_i\le\bigvee_i\nu_i\le\lambda$ implies $\lambda=\bigvee_i\nu_i$. So,

$$\tau(\lambda) = \tau\left(\bigvee_i \nu_i\right) \ge \bigwedge_i \tau(\nu_i) \ge r$$

and

$$\tau^*(\lambda) = \tau^*\left(\bigvee_i \nu_i\right) \le \bigvee_i \tau^*(\nu_i) \le s.$$

This is a contradiction. Hence, $\tau_{\mathcal{I}_{\delta_\tau,\delta^*_{\tau^*}}} \le \tau$ and $\tau^*_{\mathcal{I}_{\delta_\tau,\delta^*_{\tau^*}}} \ge \tau^*$. Finally, $\tau_{\mathcal{I}_{\delta_\tau,\delta^*_{\tau^*}}} = \tau$ and $\tau^*_{\mathcal{I}_{\delta_\tau,\delta^*_{\tau^*}}} = \tau^*$ is obtained.

(5) For every double fuzzy topology (τ,τ^*) on X, since M is a completely distributive lattice, by (2), $(\delta_\tau,\delta^*_{\tau^*})$ is principal. By Theorem 3.6, the pair $(\tau_{\delta_\tau},\tau^*_{\delta^*_{\tau^*}})$ is a double fuzzy topology on X. Since

$$\delta_\tau(\lambda,\lambda\mapsto\underline{0}) = \bigwedge\{\tau(\nu)\mapsto 0_M \mid \lambda\le\nu\le\lambda\} = \tau(\lambda)\mapsto 0_M$$

and

$$\delta^*_{\tau^*}(\lambda,\lambda\mapsto\underline{0}) = \bigvee\{\tau^*(\nu)\mapsto 0_M \mid \lambda\le\nu\le\lambda\} = \tau^*(\lambda)\mapsto 0_M,$$

we have

$$\tau_{\delta_\tau}(\lambda) = \delta_\tau(\lambda,\lambda\mapsto\underline{0})\mapsto 0_M = \tau(\lambda),$$

and

$$\tau^*_{\delta^*_{\tau^*}}(\lambda) = \delta^*_{\tau^*}(\lambda,\lambda\mapsto\underline{0})\mapsto 0_M = \tau^*(\lambda). \quad\square$$

Theorem 3.14. *Let (X,τ_1,τ_1^*) and (Y,τ_2,τ_2^*) be two double fuzzy topological spaces and let $\phi:(X,\tau_1,\tau_1^*)\longrightarrow(Y,\tau_2,\tau_2^*)$ be an LF-continuous map. Then $\phi:(X,\delta_{\tau_1},\delta^*_{\tau_1^*})\longrightarrow(Y,\delta_{\tau_2},\delta^*_{\tau_2^*})$ is a double fuzzy preproximally continuous map.*

Proof. Let $\nu, \rho \in L^Y$, then we have

$$\begin{aligned}\delta_{\tau_2}(\nu,\rho) &= \bigwedge\{\tau_2(\gamma)\mapsto 0_M \mid \gamma\in\Theta_{\nu,\rho}\}\\ &\geq \bigwedge\{\tau_1(\phi^{\leftarrow}(\gamma))\mapsto 0_M \mid \phi^{\leftarrow}(\gamma)\in\Theta_{\phi^{\leftarrow}(\nu),\phi^{\leftarrow}(\rho)}\}\\ &\geq \bigwedge\{\tau_1(\lambda)\mapsto 0_M \mid \lambda\in\Theta_{\phi^{\leftarrow}(\nu),\phi^{\leftarrow}(\rho)}\}\\ &= \delta_{\tau_1}(\phi^{\leftarrow}(\nu),\phi^{\leftarrow}(\rho))\end{aligned}$$

and

$$\begin{aligned}\delta^*_{\tau_2^*}(\nu,\rho) &= \bigvee\{\tau_2^*(\gamma)\mapsto 0_M \mid \gamma\in\Theta_{\nu,\rho}\}\\ &\leq \bigvee\{\tau_1^*(\phi^{\leftarrow}(\gamma))\mapsto 0_M \mid \phi^{\leftarrow}(\gamma)\in\Theta_{\phi^{\leftarrow}(\nu),\phi^{\leftarrow}(\rho)}\}\\ &\leq \bigvee\{\tau_1^*(\lambda)\mapsto 0_M \mid \lambda\in\Theta_{\phi^{\leftarrow}(\nu),\phi^{\leftarrow}(\rho)}\}\\ &= \delta^*_{\tau_1^*}(\phi^{\leftarrow}(\nu),\phi^{\leftarrow}(\rho)).\end{aligned}$$

Therefore, $\phi : (X, \delta_{\tau_1}, \delta^*_{\tau_1^*}) \longrightarrow (Y, \delta_{\tau_2}, \delta^*_{\tau_2^*})$ is a double fuzzy preproximally continuous map. □

Theorem 3.15. *Let $(X, \mathcal{I})$ be a double fuzzy interior space. Define the maps $\delta_{\mathcal{I}}, \delta^*_{\mathcal{I}} : L^X \times L^X \longrightarrow M$ as follows:*

$$\delta_{\mathcal{I}}(\lambda,\mu) = \bigwedge\{r\mapsto 0_M \mid \lambda \leq \mathcal{I}(\mu\mapsto \underline{0}, r\mapsto 0_M, s\mapsto 0_M)\}$$

and

$$\delta^*_{\mathcal{I}}(\lambda,\mu) = \bigvee\{s\mapsto 0_M \mid \lambda \leq \mathcal{I}(\mu\mapsto \underline{0}, r\mapsto 0_M, s\mapsto 0_M)\}.$$

Then we have the following properties:

(1) $(\delta_{\mathcal{I}}, \delta^*_{\mathcal{I}})$ is a double fuzzy preproximity on X.

(2) If M is an order dense chain, then $(\delta_{\mathcal{I}}, \delta^*_{\mathcal{I}})$ is principal.

(3) If M is a chain, $\mathcal{I}_{\delta_{\mathcal{I}},\delta^*_{\mathcal{I}}}(\lambda, r, s) \leq \mathcal{I}(\lambda, r, s)$, for each $\lambda \in L^X$, $r \in M_0$ and $s \in M_1$.

(4) If $\mathcal{I}$ is topological and M is idempotent, then $(\delta_{\mathcal{I}}, \delta^*_{\mathcal{I}})$ is a double fuzzy quasi-proximity on X.

Proof. (1)

(P1)

$$\begin{aligned} r\leq s\mapsto 0_M &\Rightarrow r\mapsto 0_M \geq (s\mapsto 0_M)\mapsto 0_M\\ &\Rightarrow \delta_{\mathcal{I}}(\lambda,\mu)\geq \delta^*_{\mathcal{I}}(\lambda,\mu)\mapsto 0_M.\end{aligned}$$

(P2) $\delta_{\mathcal{I}}(\underline{1},\underline{0}) = \delta_{\mathcal{I}}(\underline{0},\underline{1}) = 0_M$ and $\delta^*_{\mathcal{I}}(\underline{1},\underline{0}) = \delta^*_{\mathcal{I}}(\underline{0},\underline{1}) = 1_M$.

(P3) Let $\delta_{\mathcal{I}}(\lambda,\mu) \neq 1_M$ and $\delta^*_{\mathcal{I}}(\lambda,\mu) \neq 0_M$, then by the definitions,

$$\lambda \leq \mathcal{I}(\mu\mapsto\underline{0}, r\mapsto 0_M, s\mapsto 0_M) \leq \mu\mapsto \underline{0},$$

so $\lambda \leq \mu\mapsto\underline{0}$.

(P4) Let $\lambda_1, \lambda_2 \in L^X$ with $\lambda_1 \leq \lambda_2$, we have

$$\lambda_1 \leq \lambda_2 \leq \mathcal{I}(\mu\mapsto\underline{0}, r\mapsto 0_M, s\mapsto 0_M),$$

then, $\delta_{\mathcal{I}}(\lambda_1,\mu) \leq \delta_{\mathcal{I}}(\lambda_2,\mu)$ and $\delta^*_{\mathcal{I}}(\lambda_1,\mu) \geq \delta^*_{\mathcal{I}}(\lambda_2,\mu)$.

(P5) Since

$$\lambda_1 \leq \mathcal{I}(\mu_1\mapsto\underline{0}, r_1\mapsto 0_M, s_1\mapsto 0_M)$$

and

$$\lambda_2 \leq \mathcal{I}(\mu_2\mapsto\underline{0}, r_2\mapsto 0_M, s_2\mapsto 0_M),$$

imply

$$\lambda_1\odot\lambda_2 \leq \mathcal{I}((\mu_1\oplus\mu_2)\mapsto\underline{0}, (r_1\mapsto 0_M)\odot(r_2\mapsto 0_M), (s_1\mapsto 0_M)\oplus(s_2\mapsto 0_M)),$$

we have

$$\begin{aligned}
\delta_{\mathcal{I}}(\lambda_1,\mu_1)\oplus\delta_{\mathcal{I}}(\lambda_2,\mu_2) &= \bigwedge\{r_1\mapsto 0_M \mid \lambda_1\le \mathcal{I}(\mu_1\mapsto \underline{0}, r_1\mapsto 0_M, s_1\mapsto 0_M)\}\\
&\quad \oplus\bigwedge\{r_2\mapsto 0_M \mid \lambda_2\le \mathcal{I}(\mu_2\mapsto \underline{0}, r_2\mapsto 0_M, s_2\mapsto 0_M)\}\\
&= \bigwedge\{\oplus_i(r_i\mapsto 0_M)\mid \lambda_i\le \mathcal{I}(\mu_i\mapsto \underline{0}, r_i\mapsto 0_M, s_i\mapsto 0_M), i=1,2\}\\
&= \bigwedge\{(\odot_i r_i)\mapsto 0_M\mid \odot_i\lambda_i\le \mathcal{I}((\oplus_i\mu_i)\mapsto \underline{0}, (\oplus_i r_i)\mapsto 0_M, (\odot_i s_i)\mapsto 0_M)\}\\
&\ge \bigwedge\{(\oplus_i r_i)\mapsto 0_M\mid \odot_i\lambda_i\le \mathcal{I}((\oplus_i\mu_i)\mapsto \underline{0}, (\oplus_i r_i)\mapsto 0_M, (\odot_i s_i)\mapsto 0_M)\}\\
&\ge \bigwedge\{t\mapsto 0_M\mid \lambda_1\odot\lambda_2\le \mathcal{I}((\mu_1\oplus\mu_2)\mapsto \underline{0}, t\mapsto 0_M, (s_1\odot s_2)\mapsto 0_M)\}\\
&= \delta_{\mathcal{I}}(\lambda_1\odot\lambda_2,\mu_1\oplus\mu_2)\\
\delta^*_{\mathcal{I}}(\lambda_1,\mu_1)\odot\delta^*_{\mathcal{I}}(\lambda_2,\mu_2) &= \bigvee\{s_1\mapsto 0_M \mid \lambda_1\le \mathcal{I}(\mu_1\mapsto \underline{0}, r_1\mapsto 0_M, s_1\mapsto 0_M)\}\\
&\quad \odot\bigvee\{s_2\mapsto 0_M \mid \lambda_2\le \mathcal{I}(\mu_2\mapsto \underline{0}, r_2\mapsto 0_M, s_2\mapsto 0_M)\}\\
&= \bigvee\{\odot_i(s_i\mapsto 0_M)\mid \lambda_i\le \mathcal{I}(\mu_i\mapsto \underline{0}, r_i\mapsto 0_M, s_i\mapsto 0_M), i=1,2\}\\
&= \bigvee\{(\oplus_i s_i)\mapsto 0_M\mid \odot_i\lambda_i\le \mathcal{I}((\oplus_i\mu_i)\mapsto \underline{0}, (\oplus_i r_i)\mapsto 0_M, (\odot_i s_i)\mapsto 0_M)\}\\
&\le \bigvee\{(\odot_i s_i)\mapsto 0_M\mid \odot_i\lambda_i\le \mathcal{I}((\oplus_i\mu_i)\mapsto \underline{0}, (\oplus_i r_i)\mapsto 0_M, (\odot_i s_i)\mapsto 0_M)\}\\
&\le \bigvee\{k\mapsto 0_M\mid \lambda_1\odot\lambda_2\le \mathcal{I}((\mu_1\oplus\mu_2)\mapsto \underline{0}, (\oplus_i r_i)\mapsto 0_M, k\mapsto 0_M)\}\\
&= \delta^*_{\mathcal{I}}(\lambda_1\odot\lambda_2,\mu_1\oplus\mu_2).
\end{aligned}$$

(2) Suppose there exists a family $\{\nu_i\}_{i\in\Gamma}$ such that $\delta_{\mathcal{I}}\left(\bigvee_{i\in\Gamma}\nu_i,\lambda\right)\not\le\bigvee_{i\in\Gamma}\delta_{\mathcal{I}}(\nu_i,\lambda)$ and $\delta^*_{\mathcal{I}}\left(\bigvee_{i\in\Gamma}\nu_i,\lambda\right)\not\ge\bigwedge_{i\in\Gamma}\delta^*_{\mathcal{I}}(\nu_i,\lambda)$. Since M is an order dense chain, there exist $r\in M_0$, $s\in M_1$ such that

$$\delta_{\mathcal{I}}\left(\bigvee_{i\in\Gamma}\nu_i,\lambda\right)>r>\bigvee_{i\in\Gamma}\delta_{\mathcal{I}}(\nu_i,\lambda),\qquad \delta^*_{\mathcal{I}}\left(\bigvee_{i\in\Gamma}\nu_i,\lambda\right)<s<\bigwedge_{i\in\Gamma}\delta^*_{\mathcal{I}}(\nu_i,\lambda).$$

Since $\delta_{\mathcal{I}}(\nu_i,\lambda)<r$ and $\delta^*_{\mathcal{I}}(\nu_i,\lambda)>s$ for every $i\in\Gamma$, there exists $r_i,s_i\in M$ such that $r_i\mapsto 0_M<r$, $s_i\mapsto 0_M>s$ with $\nu_i\le\mathcal{I}(\lambda\mapsto\underline{0}, r_i\mapsto 0_M, s_i\mapsto 0_M)$. Put $t=\bigvee_{i\in\Gamma}r_i$, $k=\bigwedge_{i\in\Gamma}s_i$. Then $\bigvee_{i\in\Gamma}\nu_i\le\mathcal{I}(\lambda\mapsto\underline{0}, t\mapsto 0_M, k\mapsto 0_M)$, so

$$\delta_{\mathcal{I}}\left(\bigvee_{i\in\Gamma}\nu_i,\lambda\right)\le t\mapsto 0_M<r,\qquad \delta^*_{\mathcal{I}}\left(\bigvee_{i\in\Gamma}\nu_i,\lambda\right)\ge k\mapsto 0_M>s.$$

This is a contradiction.

(3) Let M be a chain. Since

$$\mathcal{I}_{\delta_{\mathcal{I}},\delta^*_{\mathcal{I}}}(\lambda,r,s)=\bigvee\{\rho_i\mid \delta_{\mathcal{I}}(\rho_i,\lambda\mapsto\underline{0})<r\mapsto 0_M \text{ and } \delta^*_{\mathcal{I}}(\rho_i,\lambda\mapsto\underline{0})>s\mapsto 0_M\}$$

for each $i\in\Gamma$, there exists $r_i,s_i\in M$ such that $r_i>r$ and $s_i<s$ with $\rho_i\le\mathcal{I}(\lambda,r_i,s_i)$. Put $t=\bigwedge_{i\in\Gamma}r_i$, $k=\bigvee_{i\in\Gamma}s_i$. Then $\bigvee_{i\in\Gamma}\rho_i\le\mathcal{I}(\lambda,t,k)$. Hence,

$$\mathcal{I}_{\delta_{\mathcal{I}},\delta^*_{\mathcal{I}}}(\lambda,r,s)\le\mathcal{I}(\lambda,t,k)\le\mathcal{I}(\lambda,r,s).$$

(4) Since $\lambda\le\mathcal{I}(\mu\mapsto\underline{0}, r\mapsto 0_M, s\mapsto 0_M)$ implies $\lambda\le\mathcal{I}(\mathcal{I}(\mu\mapsto\underline{0}, r\mapsto 0_M, s\mapsto 0_M), r\mapsto 0_M, s\mapsto 0_M)$ and $\mathcal{I}(\mu\mapsto\underline{0}, r\mapsto 0_M, s\mapsto 0_M)=\mathcal{I}(\mu\mapsto\underline{0}, r\mapsto 0_M, s\mapsto 0_M)$. Put $\rho=\mathcal{I}(\mu\mapsto\underline{0}, r\mapsto 0_M, s\mapsto 0_M)\mapsto\underline{0}$. Then

$$\delta_{\mathcal{I}}(\lambda,\mu)\ge\delta_{\mathcal{I}}(\lambda,\rho)\oplus\delta_{\mathcal{I}}(\rho\mapsto\underline{0},\mu)\ge\bigwedge_{\rho\in L^X}\{\delta_{\mathcal{I}}(\lambda,\rho)\oplus\delta_{\mathcal{I}}(\rho\mapsto\underline{0},\mu)\}.\quad\square$$

Corollary 3.16. (1) *Define* $F : \mathrm{PDFpp} \longrightarrow \mathrm{DFTop}$ *by* $F(X, \delta, \delta^*) = (X, \tau_\delta, \tau^*_{\delta^*})$ *and* $F(\phi) = \phi$. *Then* F *is a functor.*

(2) *Define* $G : \mathrm{DFpp} \longrightarrow \mathrm{DFInt}$ *by* $G(X, \delta, \delta^*) = (X, \mathcal{I}_{\delta,\delta^*})$ *and* $G(\phi) = \phi$. *Then* G *is a functor.*

(3) *Define* $H : \mathrm{DFpp} \longrightarrow \mathrm{DFCl}$ *by* $H(X, \delta, \delta^*) = (X, \mathcal{C}_{\delta,\delta^*})$ *and* $H(\phi) = \phi$. *Then* H *is a functor.*

(4) *Define* $K : \mathrm{DFTop} \longrightarrow \mathrm{DFpp}$ *by* $K(X, \tau, \tau^*) = (X, \delta_\tau, \delta^*_{\tau^*})$ *and* $K(\phi) = \phi$. *Then* K *is a functor and if* M *is completely distributive, then* $F \circ K = \mathrm{id}_{\mathrm{DFTop}}$.

Proof. (1) It follows from Theorems 3.6 and 3.7.

(2) and (3) are straightforward from Theorems 3.8 and 3.12 (4) and Theorems 3.10 and 3.12 (1), respectively.

(4) It follows from Theorems 3.13 and 3.14. □

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